Application of two spectral methods to a problem of convection with uniform internal heat source

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Abstract

Two methods based on Fourier series expansions (a Chandrasekhar functions-based method and a shifted Legendre polynomials -based method) are used to study analytically the eigenvalue problem governing the linear convection problem with an uniform internal heat source in a horizontal fluid layer bounded by two rigid walls. For each method some theoretical remarks are made. Numerical results are given and they are compared with some existing ones. Good agrement is found.

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1 Problem setting

The effects of the presence in a fluid of an internal heat source have been experimentally, numerically and analytically investigated by researchers in many convection problems [6], [7], [8],[9]. The investigations concerned the effects of the heating and cooling rate. Various conditions were imposed on the lower and upper boundaries. The motion in the

atmosphere or mantle convection are two among phenomena of natural convection induced by internal heat sources. They bifurcate from the conduction state as a result of its loss of stability. In spite of their importance, due to the occurrence of variable coefficients in the nonlinear partial differential equations governing the evolution of the perturbations around the basic equilibrium, so far these phenomena were treated mostly numerically and experimentally.

Herein a horizontal layer of viscous incompressible fluid with constant viscosity and thermal conductivity coefficients ν and k is considered [9]. In this context, the heat and hydrostatic transfer equations are [9]

$$\eta = k\frac{\partial^2\theta_B}{\partial z^2}, \tag{1}$$

$$\frac{dp_B}{dz} = -\rho_B g, \tag{2}$$

where $\eta = const.$ is the heating rate, θ_B, p_B and ρ_B are the potential temperature, pressure and density in the basic state, respectively. In the fluid, the temperature at all point varies at the same rate as the boundary temperature, so the problem is characterized by a constant potential temperature difference between the lower and the upper boundaries $\Delta\theta_B = \theta_{B_0} - \theta_{B_1}$. Taking into account (1) this leads to the following formula for the potential temperature distribution [9]

$$\theta_B = \theta_{B_0} - \frac{\Delta\theta_B}{h}\Big(z + \frac{h}{2}\Big) + \frac{\eta}{2k}\Big[z^2 - \Big(\frac{h^2}{2}\Big)^2\Big]. \tag{3}$$

In nondimensional variables the system of equations characterizing the problem is

$$\begin{cases} \dfrac{d\mathbf{U}}{dt} = -\nabla p' + \Delta\mathbf{U} + Gr\theta'\mathbf{k}, \\ \\ \mathrm{div}\mathbf{U} = 0, \\ \\ \dfrac{d\theta'}{dt} = (1 - Nz)\mathbf{U}\mathbf{k} + Pr^{-1}\Delta\theta', \end{cases} \tag{4}$$

where $\mathbf{U} = (u, v, w)$ is the velocity, θ' and p' are the temperature and pressure deviations from the basic state [9], Gr is the Grashof number,

Pr is the Prandtl number and N is a nondimensional parameter characterizing the heating (cooling) rate of the layer.

The boundaries are assumed rigid and ideal heat conducting, so the boundary conditions read

$$\mathbf{U} = \theta' = 0 \text{ at } z = -\frac{1}{2} \text{ and } z = \frac{1}{2}. \tag{5}$$

In [9] the numerical investigations concerned the vertical distribution of the total heat fluxes and their individual components for small and moderate supercritical Rayleigh number in the presence of a uniform heat source.

The eigenvalue problem associated with the equations for a convection problem with an uniform internal heat source in a horizontal fluid layer bounded by two rigid walls was deduced in [2].

Consider the viscous incompressible fluid confined into a periodicity rectangular box $V : 0 \leq x \leq a_1$, $0 \leq y \leq a_2$, $-\frac{1}{2} \leq z \leq \frac{1}{2}$ [4] bounded by two rigid horizontal walls. The corresponding eigenvalue problem [2] has the form

$$\begin{cases} (D^2 - a^2)^2 W - a^2 Ra\Theta = 0, \\ (D^2 - a^2)\Theta + (1 - Nz)W = 0. \end{cases} \tag{6}$$

with the boundary conditions

$$W = DW = \Theta = 0 \text{ at } z = \pm\frac{1}{2}. \tag{7}$$

In (6) the Rayleigh number Ra represents the eigenvalue while (W, Θ) represents the corresponding eigenvector. The analytical study of this stability problem consists in finding the smallest eigenvalue, i.e. the critical value of the Rayleigh number at which the convection sets in.

In [2] the analytical study of the eigenvalue problem (6)-(7) was performed by means of a method from [1]. First the system (6)-(7) was written in a more convenient independent variable $x = z + \frac{1}{2}$. Then, two methods (one based on Fourier series expansions of the unknown functions and other a variational one) were used in order to find the smallest eigenvalue. Here, the analytical study in also based on Fourier series expansions of the unknown functions, but the expansion functions satisfy all boundary conditions.

Taking into account the form of the boundary conditions two methods are used and, for each of them, some analytical remarks on the chosen sets of expansion functions are presented.

2 A method based on Chandrasekhar functions

In this method, the unknown function W is expanded upon a complete set of orthogonal functions that satisfy all boundary conditions $\left(W = DW = 0 \text{ at } z = \pm\frac{1}{2}\right)$ and then, from $(6)_2$ we find the expression of the unknown function Θ. Replacing these expansions in $(6)_1$ and imposing the condition that the left-hand side of the obtained equation to be orthogonal to each function from the expansion set, we obtain an algebraic system of equations which leads us to the secular equation, yielding the critical value of the Rayleigh number.

When the normal component of the velocity and its derivative are zero at $z = -\frac{1}{2}$ and $z = \frac{1}{2}$, the classical set of complete orthogonal functions that satisfy these conditions are the Chandrasekhar sets of functions $\{C_n\}_{n\in\mathbb{N}}$, $\{S_n\}_{n\in\mathbb{N}}$[1]

$$C_n(z) = \frac{\cosh \lambda_n z}{\cosh \lambda_n/2} - \frac{\cos \lambda_n z}{\cos \lambda_n/2}, \tag{8}$$

$$S_n(z) = \frac{\sinh(\mu_n z)}{\sinh(\mu_n/2)} - \frac{\sin(\mu_n z)}{\sin(\mu_n/2)} \tag{9}$$

where λ_n and μ_n are the positive roots of the equations $\tanh\left(\frac{\lambda}{2}\right) + \tan\left(\frac{\lambda}{2}\right) = 0$ and $\coth\left(\frac{\mu}{2}\right) - \cot\left(\frac{\mu}{2}\right) = 0$. We have

$$\int_{-0.5}^{0.5} C_n(z)C_m(z)dz = \int_{-0.5}^{0.5} S_n(z)S_m(z)dz = \delta_{mn}.$$

By definition, the functions C_n and S_n and their derivatives vanish at $x = \pm\frac{1}{2}$ so the boundary conditions (7) are satisfied.

Let us consider $W=\sum\limits_{n=1}^{\infty}W_nC_n(z)$. From $(6)_2$ we obtain the expression of the unknown function Θ,

$$\Theta=A\cosh az+B\sinh az+\frac{W_n\cosh\lambda_n z(Nz-1)}{(\lambda_n^2-a^2)\cosh\lambda_n/2}-\frac{2\lambda_n NW_n}{(\lambda_n^2-a^2)^2\cosh\lambda_n/2}\cdot$$
$$\cdot\sinh\lambda_n z+\frac{(1-Nz)W_n\cos\lambda_n z}{(\lambda_n^2+a^2)\cos\lambda_{n/2}}-\frac{2\lambda_n NW_n}{(\lambda_n^2+a^2)^2\cos\lambda_n/2}\sin\lambda_n z,$$

where $A=\dfrac{2a^2W_n}{(\lambda_n^2-a^2)(\lambda_n^2+a^2)\cosh a/2}$ and

$$B=\frac{8\lambda_n^3NW_na^2}{(\lambda_n^2-a^2)^2(\lambda_n^2+a^2)^2\cosh\lambda_n/2}-\frac{a^2NW_n}{(\lambda_n^2-a^2)(\lambda_n^2+a^2)}.$$

However, in our case, replacing these expressions in $(6)_1$ and imposing the condition that the left-hand side of the obtained equation to be orthogonal to C_m, $m\in\mathbb{N}$, we obtain an expression in which the physical parameter N is missing. The mathematical explanation is that the chosen set of expansion functions introduced an extraparity (inexistent in the given problem), leading to the loss of one of the physical parameter, in this case the cooling (heating) rate N.

Remark. The physical parameter N also disappear when the expansion functions are S_n, $n=1,2,....$

Another explanation could be the fact that we have no physical or mathematical reason to assume that W is either even or odd. The general form of W will be considered elsewhere.

3 A method based on shifted Legendre polynomials

In order to avoid the loss of N, we use a different set of orthogonal functions, namely a basis of shifted Legendre polynomials (SLP) on $[0,1]$.

Let us modify the system (6) by a translation of the variable z, $x=z+\dfrac{1}{2}$, such that the eigenvalue problem becomes

$$\begin{cases}(D^2-a^2)^2W-a^2Ra\Theta=0,\\(D^2-a^2)\Theta+(N_1-Nx)W=0,\end{cases}\tag{10}$$

with $N_1 = 1 + \dfrac{N}{2}$ and the boundary conditions

$$W = DW = \Theta = 0 \text{ at } x = 0 \text{ and } 1. \tag{11}$$

Starting with the classical Legendre polynomials defined on $(-1, 1)$, let us introduce the complete sets of expansion functions. We are interested in expansion functions that satisfy all boundary conditions. Let $H_0^1(0, 1)$, $H_0^2(0, 1)$ be two Hilbert spaces [5]

$$H_0^1(0,1) = \{f | f, f' \in L^2(0,1), f(0) = f(1) = 0\},$$

$$H_0^2(0,1) = \{f | f, f', f'' \in L^2(0,1), f(0) = f(1) = f'(0) = f'(1) = 0\}$$

and denote by L_k the Legendre polynomials defined on $(-1, 1)$. By means of them, we construct the SLP (denoted by us by Q_k) on (a, b), namely $Q_k(x) = L_k\Big(\dfrac{2x - a - b}{b - a}\Big)$. Taking $(a, b) = (0, 1)$ we find that Q_k are orthogonal polynomials on the interval $(0, 1)$, i.e. $\displaystyle\int_0^1 Q_i Q_j dz = \frac{1}{2i+1}\delta_{ij}$. Using the identity [5]

$$2(2i + 1)Q_i(z) = Q'_{i+1}(z) - Q'_{i-1}(z). \tag{12}$$

we define the complete sets of orthogonal functions $\{\phi_i\}_{i=1,2,\ldots} \subset H_0^1(0, 1)$,

$$\phi_i(z) = \int_0^z Q_i(t)dt = \frac{Q_{i+1} - Q_{i-1}}{2(2i+1)},$$

satisfying boundary conditions $\phi_i(0) = \phi_i(1) = 0$ at $z = 0$ and 1 and $\{\beta_i\}_{i=1,2,\ldots} \subset H_0^2(0, 1)$,

$$\beta_i(z) = \int_0^z \int_0^s Q_{i+1}(t)dtds = \frac{1}{4}\Big[\frac{Q_{i+3} - Q_{i+1}}{(2i+3)(2i+5)} - \frac{Q_{i+1} - Q_{i-1}}{(2i+1)(2i+3)}\Big],$$

satisfying boundary conditions $\beta_i(0) = \beta_i(1) = \beta'_i(0) = \beta'_i(1) = 0$ at $z = 0$ and 1.

Remark. We could also work with SLP on $(a, b) = \Big(-\dfrac{1}{2}, \dfrac{1}{2}\Big)$. However, the choice $(a, b) = (0, 1)$ leads us to simplified numerical evaluations.

The system (6) can be solved numerically by approximating the solution (W, Θ) by

$$W = \sum_{i=1}^{n} W_i \beta_i(z), \quad \Theta = \sum_{i=1}^{n} \Theta_i \phi_i(z) \tag{13}$$

with W_i and Θ_i the Fourier coefficients. In this way, the system (6) can be written in terms of the expansion functions only

$$\begin{cases} \sum\limits_{i=1}^{n} [W_i (D^2 - a^2)^2 \beta_i - a^2 Ra \Theta_i \phi_i] = 0, \\ \sum\limits_{i=1}^{n} [\Theta_i (D^2 - a^2) \phi_i + (N_1 - Nz) W_i \beta_i] = 0. \end{cases} \tag{14}$$

Multiplying the system (14) by the vector (β_k, ϕ_k) we obtain the algebraic system

$$\begin{cases} \sum\limits_{i=1}^{n} [W_i \Big((D^2 - a^2)^2 \beta_i, \beta_k \Big) - a^2 Ra \Theta_i (\phi_i, \beta_k)] = 0, \\ \sum\limits_{i=1}^{n} [\Theta_i \Big((D^2 - a^2) \phi_i, \phi_k \Big) + W_i N_1 (\beta_i, \phi_k) - W_i N (z \beta_i, \phi_k)] = 0. \end{cases} \tag{15}$$

Taking into account the fact that the coefficients W_i, Θ_i are not all null, i.e. the Cramer determinant vanishes, the secular equation has the form

$$\begin{vmatrix} ((D^2 - a^2)^2 \beta_i, \beta_k) & -a^2 Ra (\phi_i, \beta_k) \\ N_1(\beta_i, \phi_k) - N(z\beta_i, \phi_k) & ((D^2 - a^2)\phi_i, \phi_k) \end{vmatrix} = 0. \tag{16}$$

The scalar products from (16) are given in the Appendix.

The system (10) has variable coefficients (functions of x). In this case, the following recurrence relation was used for the numerical study

$$2zQ_i = \frac{i+1}{2i+1} Q_{i+1} + Q_i + \frac{i}{2i+1} Q_{i-1}. \tag{17}$$

4 Numerical results

Taking $n = m = 1$ we obtained a first approximation of the Rayleigh number, which proved to be a good approximation compared to the one obtained in [2]. The obtained numerical results are presented in Table 1

in comparison with the results from [2]. The disadvantage of this method is given by the fact that the approximations are limited by the difficult evaluation of the associated matrix for a large number of functions in the expansion sets. However, the expressions of the neutral manifolds are easy to obtain with this method. For a large number of terms in the Fourier series expansions, we must use an algorithm for solving the algebraic equation (16). For instance in [5], the Arnoldi algorithm is used.

N	a^2	$Ra - Fourier$	$R_a - var.meth.$	$Ra - Legendre$
0	9.711	1715.079324	1749.97575	1749.95727
1	9.711	1711.742588	1746.804944	1746.809422
2	9.711	1701.891001	1737.45025	1737.450242
1	10.0	1712.257687	1747.29100	1747.290998
4	10.0	1664.341789	1701.62704	1701.627037
4	12.0	1685.422373	1723.62407	1723.624047
8	12.0	1547.460446	1590.19681	1590.196769
9	12.0	1508.147637	1551.72378	1551.723746
10	12.0	1468.449223	1512.69203	1512.691998
12	12	1389.837162	1434.90396	1434.903926
16	12	1243.442054	1288.50149	1288.501459
10	9.0	1482.527042	1525.59302	1525.593072
11	9.0	1446.915467	1490.55802	1490.558078
12	9.00	1411.401914	1455.48233	1455.482384

Table 1. Numerical evaluations of the Rayleigh number for various values of the parameters N and a.

When the wavenumber is kept constant an increase in the heating (cooling) rate parameter leads to a decreasing of the Rayleigh number. When $N = 0$ the problem reduces to the particular case of Rayleigh-Bénard convection and the numerical evaluation lead us to a value similar to the classical value for the Rayleigh number, i.e. $Ra = 1749.95727$ for $a = 3.117$.

5 Appendix

Let us give the expressions of the scalar products occurring in (16). Since in $(10)_1$ the expression $((D^2-a^2)^2\beta_i,\beta_k)$ is written as

$$((D^2-a^2)^2\beta_i,\beta_k) = (D^4\beta_i,\beta_k) - 2a^2(D^2\beta_i,\beta_k) + a^4(\beta_i,\beta_k)$$

let us simplify these products or simply evaluate them. Taking into account the definition of the scalar product on $L^2(0,1)$, i.e. $(f,g) = \int_0^1 fg dz$ and the boundary conditions satisfied by the expansion functions, we have

$$(D^4\beta_i,\beta_k) = (\beta_i'',\beta_k'') = \begin{cases} \dfrac{1}{2i+3} \text{ if } i=k, \\ \\ 0 \text{ if } i \neq k \end{cases} \tag{18}$$

and

$$(D^2\beta_i,\beta_k) = -(\beta_i',\beta_k') = \begin{cases} -\dfrac{1}{2(2i+1)(2i+3)(2i+5)} \text{ if } i=k, \\ \\ \dfrac{1}{4(2i-1)(2i+1)(2i+3)} \text{ if } i=k+2, \\ \\ 0 \text{ otherwise} \end{cases} \tag{19}$$

Given the fact that $(\beta_i,\beta_k) = \dfrac{1}{2}\Big(\dfrac{\phi_{i+2}-\phi_i}{2i+3}, \dfrac{\phi_{k+2}-\phi_k}{2k+3}\Big)$ we first evaluated the product (ϕ_i,ϕ_k) and we get

$$(\phi_i,\phi_k) = \begin{cases} \dfrac{1}{2(2i-1)(2i+1)(2i+3)} \text{ if } i=k, \\ \\ -\dfrac{1}{4(2i+1)(2i+3)(2i+5)} \text{ if } i=k-2, \\ \\ 0 \text{ otherwise} \end{cases} \tag{20}$$

Using (20) we have

$$(\beta_i,\beta_k) = \begin{cases} \dfrac{3}{8(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)} \text{ if } i=k, \\ -\dfrac{1}{4(2i+1)(2i+3)(2i+5)(2i+7)(2i+9)} \text{ if } i=k-2, \\ \dfrac{1}{16(2i+3)(2i+5)(2i+7)(2i+9)(2i+11)} \text{ if } i=k-4, \\ 0 \text{ otherwise} \end{cases} \tag{21}$$

We also used (20) to deduce (ϕ_i,β_k), i.e.

$$(\phi_i,\beta_k) = \frac{1}{2(2k+3)}[(\phi_i,\phi_{k+2}) - (\phi_i,\phi_k)] = \\ = \begin{cases} -\dfrac{3}{8(2i-1)(2i+1)(2i+3)(2i+5)} \text{ if } i=k, \\ \dfrac{3}{8(2i-3)(2i-1)(2i+1)(2i+3)} \text{ if } i=k+2, \\ \dfrac{1}{8(2i+1)(2i+3)(2i+5)(2i+7)} \text{ if } i=k-2, \\ -\dfrac{1}{8(2i-5)(2i-3)(2i-1)(2i+1)} \text{ if } i=k+4, \\ 0 \text{ otherwise} \end{cases} \tag{22}$$

Let us remark that $(\beta_i,\phi_k) = (\phi_k,\beta_i)$.

The computations of $(D^2\phi_i,\phi_k)$ was simplified by the expressions of the ϕ_i functions. We have

$$(D^2\phi_i,\phi_k) = -(Q_i,Q_k) = \begin{cases} -\dfrac{1}{2i+1} \text{ if } i=k, \\ 0 \text{ otherwise} \end{cases}. \tag{23}$$

All the obtained expressions (18) - (24) are based on the orthogonality relationship between the SLP. In deducing the expression below we also

used the recurrence relation (17)

$$
(z\beta_i, \phi_k) = \begin{cases}
-\dfrac{i+4}{16(2i+3)(2i+5)(2i+7)(2i+9)(2i+11)} \text{ if } i = k-5, \\[2ex]
-\dfrac{1}{16(2i+3)(2i+5)(2i+7)(2i+9)} \text{ if } i = k-4, \\[2ex]
\dfrac{1}{16(2i+1)(2i+3)(2i+5)(2i+9)} \text{ if } i = k-3, \\[2ex]
\dfrac{3}{16(2i+1)(2i+3)(2i+5)(2i+7)} \text{ if } i = k-2, \\[2ex]
-\dfrac{3}{16(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)} \text{ if } i = k-1, \\[2ex]
-\dfrac{3}{16(2i-1)(2i+1)(2i+3)(2i+5)} \text{ if } i = k, \\[2ex]
-\dfrac{1}{16(2i-3)(2i+1)(2i+3)(2i+5)} \text{ if } i = k+1, \\[2ex]
\dfrac{1}{16(2i-3)(2i-1)(2i+1)(2i+3)} \text{ if} i = k+2, \\[2ex]
\dfrac{i+1}{16(2i-5)(2i-3)(2i-1)(2i+1)(2i+3)} \text{ if } i = k+3 \\[2ex]
0 \text{ otherwise}
\end{cases} \tag{24}
$$

6 Conclusions

In this paper we performed an analytical study of the eigenvalue problem corresponding to a convection problem with uniform internal heat source. We pointed out some aspects of the spectral methods that we employed concerning the sets of the expansion functions that can be used to an analytical study of this problem. As in this case the expansion sets of Chandrasekhar functions introduced an extraparity they were not ap-

propriate. However, for some other problems [3] their use proved to be successful. The method based on SLP lead to good numerical approximations. All numerical results obtained with this method are compared with the existing ones. The effect of the heating (cooling) rate on the values of the Rayleigh number is pointed out.

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